Non-transitivity in Tournament

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Definitions

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Note

The convention is to draw an arrow away from the winner and toward the loser.

Example

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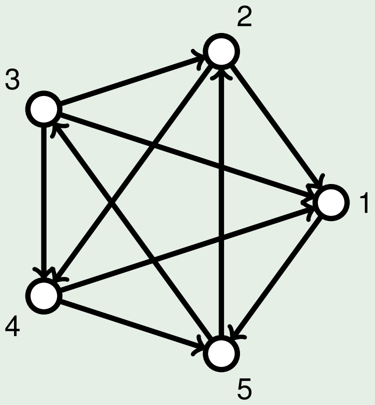

Tournament

Example

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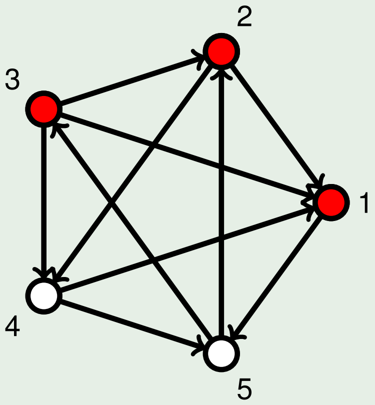

Transitive

Example

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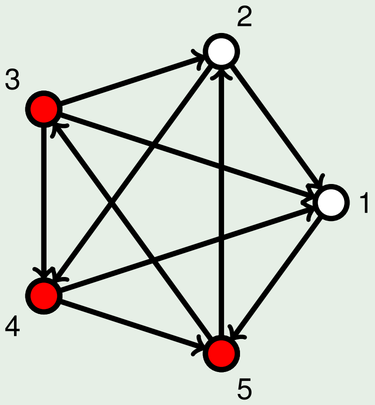

Non-transitive

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Consider 3 players. There are 3 games between them.

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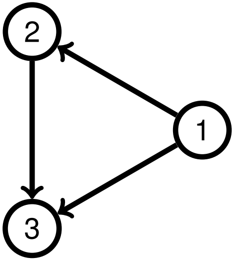

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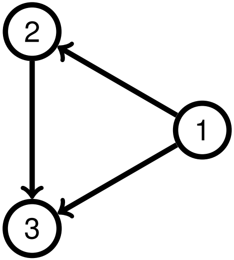
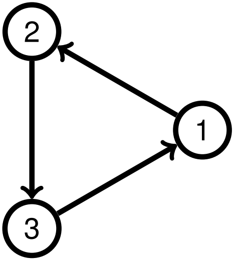


If no player wins twice, then the triple is 3-cycle.



Non-transitivity

Definition

λ is the number of 3-cycles. This measures non-transitivity of the tournament.

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Question

How big (small) can λ be?

Lower Bound

λ can be as small as 0.
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Lemma

The following statements are equivalent:

- *T is transitive.*
- *T has no cycles.*
- *The scores (# of wins) are* $\{0, 1, 2, \cdots, n-1\}$

Calculating non-transitivity

Theorem

Let w_i be the number of wins for the player i. Then

$$\lambda = \left(\sum_{i=1}^{n} (i-1)^2 - \sum_{i=1}^{n} w_i^2 \right) / 2$$

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Calculating non-transitivity

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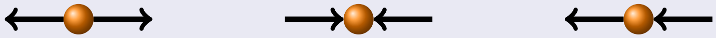
Proof.

We consider elbows, which consists of one player and two of the games played by the player. There are three types of elbows.



Calculating non-transitivity

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1. Sum the number of broken elbows for all players. Player i has w_i wins and $(n-1) - w_i$ loses, so player i has $w_i(n-1-w_i)$ broken elbows.

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2. Each elbow comes from an unique triple. If a triple is transitive, it has one broken elbow. If a triple is a 3-cycle, it has 3 broken elbows. Therefore, there are $(\binom{n}{3}-\lambda)+3\lambda$ broken elbows.

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Therefore, $\sum_{i=1}^{n} w_i(n-1-w_i) = (\binom{n}{3}-\lambda)+3\lambda.$ □

Upper Bound

Corollary

If n is odd, the maximum occurs when $w_i = (n-1)/2$ for all players. $\lambda = (n^3 - n)/24$.

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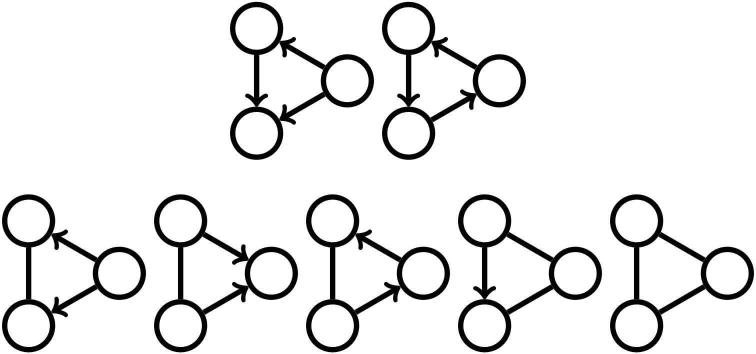
If n is even, the maximum occurs when $w_i = n/2$ for $n/2$ players and $w_i = (n-2)/2$ for other $n/2$ players. $\lambda = (n^3 - 4n)/24$.

Triples with draws

Let's allow draws. There are 7 types of triples now.



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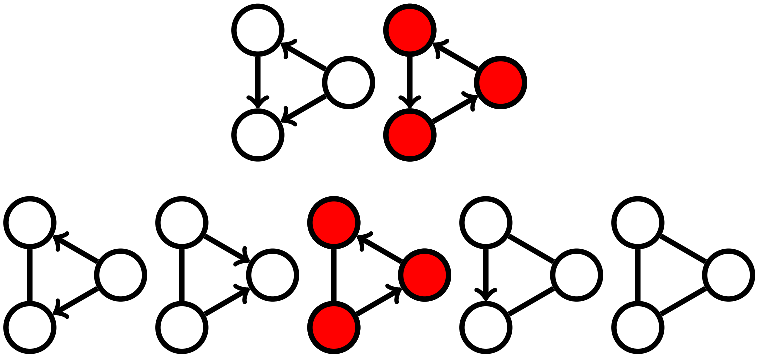

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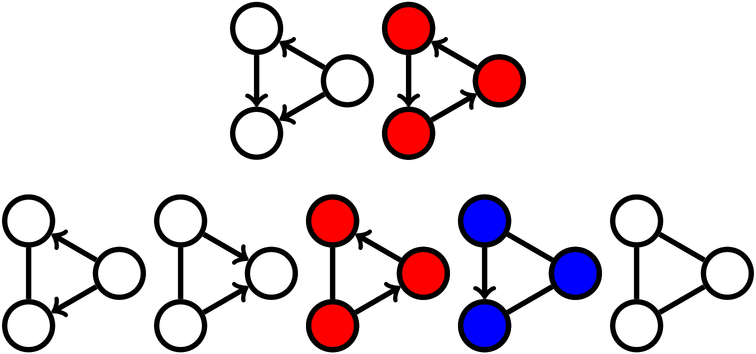
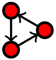
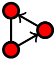
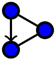
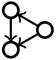
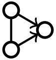
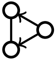
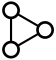


Elbows with draws

							
←○←	3	1	0	1	0	0	0
←○→	0	0	0	1	0	1	0
→○←	0	0	0	1	1	0	0
—○→	0	1	1	0	2	0	0
→○—	0	1	1	0	0	2	0
—○—	0	0	1	0	0	0	3
number	x	y	z	a	b	c	d

Non-transitivity

elbows	numbers
$\leftarrow\!\circ\!\leftarrow$	$3x + y + a$
$\leftarrow\!\circ\!\rightarrow$	$a + c$
$\rightarrow\!\circ\!\leftarrow$	$a + b$
$-\!\circ\!\rightarrow$	$y + z + 2b$
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$$4(3a + y + a) - 2(a + c) - 2(a + b) + (y + z + 2b) + (y + z + 2c)$$
$$= 12(x + \frac{1}{2}y + \frac{1}{6}z)$$

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$$4(3a + y + a) - 2(a + c) - 2(a + b) + (y + z + 2b) + (y + z + 2c) = 12(x + \frac{1}{2}y + \frac{1}{6}z)$$

Therefore, define $\lambda = x + \frac{1}{2}y + \frac{1}{6}z$.

Calculating non-transitivity

Theorem

Let w_i, d_i, l_i be the number of wins, draws, loses for the player i, respectively. Then

$$\lambda = \frac{n(n-1)(n+1)}{24} - \frac{1}{24}\sum_{i=1}^{n} d_i(d_i+2) - \frac{1}{8}\sum_{i=1}^{n}(w_i - l_i)^2$$

Proof

Proof.

We come back to the previous table.

elbows	numbers	
←○←	$3x + y + a$	$= \sum w_i l_i$
←○→	$a + c$	$= \sum \binom{w_i}{2}$
→○←	$a + b$	$= \sum \binom{l_i}{2}$
—○→	$y + z + 2b$	$= \sum w_i d_i$
→○—	$y + z + 2c$	$= \sum d_i l_i$
—○—	$z + 3d$	$= \sum \binom{d_i}{2}$

Proof

Proof (cont.)

$$\begin{aligned} 4(3a+y+a)-2(a+c)-2(a+b) \\ +(y+z+2b)+(y+z+2c) \\ =12(x+\frac{1}{2}y+\frac{1}{6}z) \end{aligned}$$

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$$\begin{aligned} 4(3a + y + a) - 2(a + c) - 2(a + b) \\ +(y + z + 2b) + (y + z + 2c) \\ = 12(x + \frac{1}{2}y + \frac{1}{6}z) \end{aligned}$$

$$4\sum w_i l_i - 2\sum \binom{w_i}{2} - 2\sum \binom{l_i}{2} + \sum w_i d_i + \sum d_i l_i = 12\lambda$$

□

Bound

Corollary

The upper and lower bound for λ is same as the case without draws.