Proof of Szemerédi's Theorem for Four Term Progressions

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We will prove Szemerédi's theorem for four term progressions. Below is the statement of the theorem.

Given $\delta > 0$, there exists $N = N(\delta)$ such that any set $A \subset [1, N]$ with $|A| \geq \delta N$ contains a non-trivial four term arithmetic progression.

Our proof for Roth's theorem, which is Szemerédi's theorem for three term progressions, can not be generalized easily. So we will look at Gowers' approach that can be generalized to k term progressions.

1. Gowers Uniformity Norm

Definition

Gowers inner product of 2^d functions is defined as follows:

$$\left\langle (f_\omega)_{\omega \in \{0,1\}^d} \right\rangle_{U^d} := E_{x,h_1,h_2,\cdots,h_d} \prod_{\omega \in \{0,1\}^d} C^{|\omega|} f_\omega(x + w \cdot h) = \frac{1}{N^{d+1}} \sum_{x,h_1,h_2,\cdots,h_d} \prod_{\omega \in \{0,1\}^d} C^{|\omega|} f_\omega(x + w \cdot h)$$

where $f_\omega : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, C is the conjugation operator (i.e. $Cf = \overline{f}$), $|\omega|$ is the number of 1 in ω, and $\omega \cdot h = \sum_{k=1}^{d} \omega_k h_k$ where $\omega = (\omega_1, \omega_2, \cdots, \omega_d)$ and $h = (h_1, h_2, \cdots, h_d)$.

Example

$\langle f_0, f_1 \rangle_{U^1} = E_{x,h} f_0(x) \overline{f_1(x+h)} = E_x f_0(x) \cdot E_x \overline{f_1(x)}$

$\langle f_{00}, f_{10}, f_{01}, f_{11} \rangle_{U^2} = E_{x,h_1,h_2} f_{00}(x) \overline{f_{10}(x+h_1) f_{01}(x+h_2)} f_{11}(x+h_1+h_2)$

Remark

Gowers inner product behaves like inner product.

- Has conjugate symmetry
  $\langle f_0, f_1 \rangle_{U^1} = \overline{\langle f_1, f_0 \rangle_{U^1}}$
  $\langle f_{00}, f_{10}, f_{01}, f_{11} \rangle_{U^2} = \overline{\langle f_{01}, f_{11}, f_{00}, f_{10} \rangle_{U^2}} = \overline{\langle f_{10}, f_{00}, f_{11}, f_{01} \rangle_{U^2}}$
- First function has linearity
  $\langle af_0, f_1 \rangle_{U^1} = a \langle f_0, f_1 \rangle_{U^1}$
  $\langle f_0 + g_0, f_1 \rangle_{U^1} = \langle f_0, f_1 \rangle_{U^1} + \langle g_0, f_1 \rangle_{U^1}$
  $\langle af_{00}, f_{10}, f_{01}, f_{11} \rangle_{U^2} = a \langle f_{00}, f_{10}, f_{01}, f_{11} \rangle_{U^2}$
  $\langle f_{00} + g_{00}, f_{10}, f_{01}, f_{11} \rangle_{U^2} = \langle f_{00}, f_{10}, f_{01}, f_{11} \rangle_{U^2} + \langle g_{00}, f_{10}, f_{01}, f_{11} \rangle_{U^2}$
- Has positivity and positive definiteness

Definition

Gowers uniformity norm of f is

$$\|f\|_{U^d} = \left\langle (f)_{\omega \in \{0,1\}^d} \right\rangle_{U^d}^{1/2^d}$$

Example

$\|f\|_{U^1} = \langle f, f \rangle_{U^1}^{1/2} = (E(f) \cdot \overline{E(f)})^{1/2} = |E(f)|$

$\|f\|_{U^2} = \langle f, f, f, f \rangle_{U^2}^{1/4} = E_{x,h_1,h_2} f(x) \overline{f(x+h_1) f(x+h_2)} f(x+h_1+h_2)$

Remark

Gowers uniformity norm is a norm.

- Has scalability

  $\|af\|_{U^d} = \left\langle (af)_{\omega \in \{0,1\}^d} \right\rangle_{U^d}^{1/2^d} = (|a|^{2^d} \left\langle (f)_{\omega \in \{0,1\}^d} \right\rangle_{U^d})^{1/2^d} = |a| \|f\|_{U^d}$
- Has triangle inequality
- Has positivity and positive definiteness (by the recursive definition)

Recursive Definition

Let $d \geq 0$ and $\Delta(f,h)(x) := f(x)\overline{f(x+h)}$. Then

$$\begin{aligned}
\|f\|_{U^{d+1}} &= \left(E_{x,h_1,h_2,\cdots,h_{d+1}} \prod_{\omega \in \{0,1\}^{d+1}} C^{|\omega|} f(x + w \cdot h) \right)^{1/2^{d+1}} \\
&= \left(E_{x,h_1,h_2,\cdots,h_{d+1}} \prod_{\omega \in \{0,1\}^{d}} C^{|\omega|} \left(f(x + w \cdot h) \cdot \overline{f(x + w \cdot h + h_{d+1})} \right) \right)^{1/2^{d+1}} \\
&= \left(E_{x,h_1,h_2,\cdots,h_{d+1}} \prod_{\omega \in \{0,1\}^{d}} C^{|\omega|} \Delta\left(f, h_{d+1}\right)(x + w \cdot h) \right)^{1/2^{d+1}} \\
&= \left(E_k \|\Delta\left(f, k\right)\|_{U^d}^{2^d} \right)^{1/2^{d+1}} \qquad (k = h_{d+1})
\end{aligned}$$

Gowers-Cauchy-Schwarz Inequality

$$\left|\left\langle (f_\omega)_{\omega\in\{0,1\}^d}\right\rangle_{U^d}\right| \leq \prod_{\omega\in\{0,1\}^d} \|f_\omega\|_{U^d}$$

Proof for $d=2$

$$\begin{aligned}
& |\langle (f_{00}, f_{10}, f_{01}, f_{11})\rangle_{U^2}| \\
= & \left| E_{x,h_1,h_2} f_{00}(x)\overline{f_{10}(x+h_1) f_{01}(x+h_2)} f_{11}(x+h_1+h_2)\right| \\
= & \left| E_{h_1}\left(E_x f_{00}(x)\overline{f_{10}(x+h_1)}\right)\left(E_{x+h_2}\overline{f_{01}(x+h_2)} f_{11}(x+h_1+h_2)\right)\right| \\
\leq & \left(E_{h_1}|E_x f_{00}(x)\overline{f_{10}(x+h_1)}|^2\right)^{1/2}\left(E_{h_1}|E_{x+h_2}\overline{f_{01}(x+h_2)} f_{11}(x+h_1+h_2)|^2\right)^{1/2} \quad \text{(Cauchy-Schwarz)} \\
= & \left(E_{h_1}(E_x f_{00}(x)\overline{f_{10}(x+h_1)})(E_x\overline{f_{00}(x)} f_{10}(x+h_1))\right)^{1/2}\left(E_{h_1}(E_x f_{01}(x)\overline{f_{11}(x+h_1)})(E_x\overline{f_{01}(x)} f_{11}(x+h_1))\right)^{1/2} \\
= & \left(E_{h_1}(E_x f_{00}(x)\overline{f_{10}(x+h_1)})(E_{x+h_2}\overline{f_{00}(x+h_2)} f_{10}(x+h_1+h_2))\right)^{1/2} \\
& \left(E_{h_1}(E_x f_{01}(x)\overline{f_{11}(x+h_1)})(E_{x+h_2}\overline{f_{01}(x+h_2)} f_{11}(x+h_1+h_2))\right)^{1/2} \\
= & \left(E_{x,h_1,h_2} f_{00}(x)\overline{f_{10}(x+h_1) f_{00}(x+h_2)} f_{10}(x+h_1+h_2)\right)^{1/2}\left(E_{x,h_1,h_2} f_{01}(x)\overline{f_{11}(x+h_1) f_{01}(x+h_2)} f_{11}(x+h_1+h_2)\right)^{1/2} \\
= & \left(E_{h_2}\left(E_x f_{00}(x)\overline{f_{00}(x+h_2)}\right)\left(E_{x+h_1}\overline{f_{10}(x+h_1)} f_{10}(x+h_1+h_2)\right)\right)^{1/2} \\
& \left(E_{h_2}\left(E_x f_{01}(x)\overline{f_{01}(x+h_2)}\right)\left(E_{x+h_1}\overline{f_{11}(x+h_1)} f_{11}(x+h_1+h_2)\right)\right)^{1/2} \\
\leq & \left(E_{h_2}|E_x f_{00}(x)\overline{f_{00}(x+h_2)}|^2\right)^{1/4}\left(E_{h_2}|E_{x+h_1}\overline{f_{10}(x+h_1)} f_{10}(x+h_1+h_2)|^2\right)^{1/4} \\
& \left(E_{h_2}|E_x f_{01}(x)\overline{f_{01}(x+h_2)}|^2\right)^{1/4}\left(E_{h_2}|E_{x+h_1}\overline{f_{11}(x+h_1)} f_{11}(x+h_1+h_2)|^2\right)^{1/4} \quad \text{(Cauchy-Schwarz)} \\
= & \|f_{00}\|_{U^2}\cdot\|f_{10}\|_{U^2}\cdot\|f_{01}\|_{U^2}\cdot\|f_{11}\|_{U^2}
\end{aligned}$$

□

Monotocity

$$\|f\|_{U^{d-1}} \leq \|f\|_{U^d}$$

Proof)

Let $f_\omega = \begin{cases} f & (\text{if } \omega_d = 0) \\ 1 & (\text{if } \omega_d = 1)\end{cases}$ for $\omega = (\omega_1, \omega_2, \cdots, \omega_d)$. Then

$$\|f\|_{U^{d-1}}^{2^{d-1}} = |\langle (f)_{\omega\in\{0,1\}^{d-1}}\rangle| = |\langle (f_\omega)_{\omega\in\{0,1\}^d}\rangle| \leq \prod_{\omega\in\{0,1\}^d}\|f_\omega\|_{U^d} = \|f\|_{U^d}^{2^{d-1}}$$

□

Gowers Triangle Inequality

$$\|f+g\|_{U^d} \leq \|f\|_{U^d} + \|g\|_{U^d}$$

Proof)

$$\begin{aligned}
\|f_0+f_1\|_{U^d}^{2^d} &= \langle (f_0+f_1)_{\omega\in\{0,1\}^d}\rangle_{U^d} \\
&= \sum_{I\subset\{0,1\}^d} \langle (f_{I(\omega\in I)})_{\omega\in\{0,1\}^d}\rangle_{U^d} && \text{(multi-linearity)} \\
&\leq \sum_{I\subset\{0,1\}^d} \prod_{\omega\in\{0,1\}^d} \|f_{I(\omega\in I)}\|_{U^d} && \text{(GCS)} \\
&= \prod_{\omega\in\{0,1\}^d} (\|f_0\|_{U^d} + \|f_1\|_{U^d}) \\
&= (\|f_0\|_{U^d} + \|f_1\|_{U^d})^{2^d}
\end{aligned}$$

□

Generalized von Neumann Theorem

For $i = 1, 2, 3, \cdots, n$, let f_i be a function $f_i : \mathbb{Z}/N \to \mathbb{C}$ such that $|f_i(x)| \leq 1$ for all x. $n \geq 2$ and N is relatively prime with $(n-1)!$, then

$$|E_{a,d} f_1(a) f_2(a+d) \cdots f_n(a+(n-1)d)| \leq \min_{i\in 1,2,\cdots,n} \|f_i\|_{U^{n-1}}$$

Proof)

Let's use induction. For $n = 2$, we have $|E_{a,d} f_1(a) f_2(a+d)| = |Ef_1 \cdot Ef_2| \leq |Ef_i| = \|f_i\|_{U^1}$ for $i = 1, 2$.

Suppose that the theorem is true for some $n = k \geq 2$. Then, for $n = k+1$ we have

$$\begin{aligned}
& |E_{a,d} f_1(a) f_2(a+d) \cdots f_{k+1}(a+kd)|^2 \\
\leq\ & E_a |f_1(a) \cdot E_d (f_2(a+d) \cdots f_{k+1}(a+kd))|^2 \\
\leq\ & (E_a(|f_1(a)|^2))(E_a |E_d (f_2(a+d) \cdots f_{k+1}(a+kd)|^2)) && \text{(Cauchy-Schwarz)} \\
\leq\ & E_a(|E_d f_2(a+d) \cdots f_{k+1}(a+kd)|^2) && (|f_1(x)| \leq 1) \\
=\ & E_a E_{d,e} f_2(a+d) \cdots f_{k+1}(a+kd) \overline{f_2(a+e) \cdots f_{k+1}(a+ke)} \\
=\ & E_a E_{d,c} \Delta(f_2, c)(a+d) \Delta(f_3, 2c)(a+2d) \cdots \Delta(f_{k+1}, kc)(a+kd) \\
\leq\ & E_c \|\Delta(f_i, c(i-1))\|_{U^{k-1}} && \text{(induction hypothesis)} \\
\leq\ & \left(E_c \|\Delta(f_i, c(i-1))\|_{U^{k-1}}^{1/2^{k-1}}\right)^{2^{k-1}} \left(E_a \|\Delta(f_i, c(i-1))\|_{U^{k-1}}^{1/2^{k-1}}\right)^{2^{k-1}} && \text{(Hölder's inequality)} \\
=\ & \|f_i\|_{U^k} && ((i-1, N) = 1)
\end{aligned}$$

for any $i = 2, 3, \cdots, k+1$. So we are done. □

From this theorem, we see that Gowers uniformity norm controls the distribution of functions in k term progressions.

2. Gowers U^2 norm and Roth's Theorem

Proposition

Let A be a subset of $[1, N]$ with $|A| = \delta N$. Let $f(n)$ be $\begin{cases} 1-\delta & (\text{if } n \in A) \\ \delta & (\text{if } n \notin A) \end{cases}$. If $\|f\|_{U^2} \leq \delta^3/32$, then either there exist $N^2\delta^3/32$ three term progression in A, or there exists a sub-progression of length $N/3$ on which A has density $\geq 9\delta/8$.

Proof)

Let $B = C = A \cap [N/3, 2N/3]$. If this set has cardinality at most $\delta N/4$ then either $A \cap [0, N/3]$ or $A \cap [2N/3, N]$ has cardinality at least $3\delta N/8$ and we are done. If this set has cardinality at least $\delta N/4$, then the number of proper three term progressions in A is at least

$$\sum_{a,d} 1_B(a)1_C(a+d)1_A(a+2d) = \delta \sum_{a,d} 1_B(a)1_C(a+d) + \sum_{a,d} 1_B(a)1_C(a+d)f(a+2d) \geq \frac{\delta^3}{16}N^2 - \|f\|_{U^2}N^2$$

and we are done. □

This theorem says that if the balanced function has small norm, than the set has three term progressions. If the balanced function has large norm, then we use the following proposition.

Proposition

$$\|f\|_{U^2}^4 = \frac{1}{N^4}\sum_r |\widehat{f}(r)|^4$$

Proof)

$$\begin{aligned}
\|f\|_{U^2}^4 &= E_{x,h_1,h_2} f(x)\overline{f(x+h_1)f(x+h_2)}f(x+h_1+h_2) \\
&= E_{\substack{a,b,c,d \\ a-b-c+d=0}} f(a)\overline{f(b)f(c)}f(d) \\
&= \sum_r E_{a,b,c,d} f(a)\overline{f(b)f(c)}f(d)e((-a+b+c-d)r/N) \\
&= \sum_r E_{a,b,c,d} f(a)e(-ar/N)\overline{f(b)e(-br/N)f(c)e(-cr/N)}f(d)e(-dr/N) \\
&= \sum_r (E_a f(a)e(-ar/N))(E_b\overline{f(b)e(-br/N)})(E_c\overline{f(c)e(-cr/N)})(E_d f(d)e(-dr/N)) \\
&= \frac{1}{N^4}\sum_r \widehat{f}(r)\overline{\widehat{f}(r)\widehat{f}(r)}\widehat{f}(r) \\
&= \frac{1}{N^4}\sum_r |\widehat{f}(r)|^4
\end{aligned}$$

□

So, if $\|f\|_{U^2} > \delta^3/32$, then by using Parseval's formula,

$$\delta^{12} \ll \|f\|_{U^2}^4 = \frac{1}{N^4}\sum_r |\widehat{f}(r)|^4 \leq \frac{1}{N^4}(\sum_r |\widehat{f}(r)|^2)^2 = \frac{1}{N^3}(\sum_n |f(n)|^2)(\sum_r |\widehat{f}(r)|^2) \leq \frac{1}{N^2}\max_k\{|\widehat{f}(k)|^2\}$$

and $\max_k\{|\widehat{f}(k)|\} \gg \delta^6 N$.

3. Gowers U^3 norm and four term progressions

Proposition

Let A be a subset of $[1, N]$ with $|A| = \delta N$. We view A in $\mathbb{Z}/N$ and let f denote the balanced function of A. If $\|f\|_{U^2} \leq \delta^4/144$ then either there exist $N^2\delta^4/72$ four term progressions in A, or there exists a sub-progression of length $2N/5$ on which A has density $\geq 25\delta/24$.

Proof)

Let $B = C = A \cap [2N/5, 3N/5]$. If this set has cardinality at most $\delta N/6$ then either $A \cap [0, 2N/5]$ or $A \cap [3N/5, N]$ has cardinality at least $\delta 5N/12$ and we are done. If this set has cardinality at least $\delta N/6$, then the number of proper four term progressions in A is at least

$$\begin{aligned} & \sum_{a,d} 1_B(a)1_C(a+d)1_A(a+2d)1_A(a+3d) \\ = \quad & \delta^2 \sum_{a,d} 1_B(a)1_C(a+d) + \delta \sum_{a,d} 1_B(a)1_C(a+d)f(a+2d) + \sum_{a,d} 1_B(a)1_C(a+d)1_A(a+2d)f(a+3d) \\ \geq \quad & \delta^4 N^2/36 - \delta\|f\|_{U^2}N^2 - \|f\|_{U^3}N^2 \end{aligned}$$

and we are done by using monotocity of Gowers norm. □

Let $f : \mathbb{Z}/N \to [-1, 1]$. If $\|f\|_{U^3} \geq \alpha$, by the recursive definition of U^d norm,

$$\alpha^8 \leq \|f\|_{U^3}^8 = \frac{1}{N} \sum_{k \bmod N} \|\Delta(f, k)\|_{U^2}^4$$

Suppose that $|\Delta(f, k)\|_{U^2}^4 \geq \alpha^8/2$ for at most $\alpha^8 N/2$ values of k. Since $\|f\|_{U^3} \leq 1$ and $|\Delta(f, k)\|_{U^2}^4 \leq 1$ for all k, $\sum_{k \bmod N} \|\Delta(f, k)\|_{U^2}^4 < \alpha^8/2 \cdot N + 1 \cdot \alpha^8 N/2 = N\alpha^8$, and this is a contradiction. Therefore, there are at least k such that $|\Delta(f, k)\|_{U^2}^4 \geq \alpha^8/2$. Let B be the set of such k.

For $k \in B$, we have

$$\begin{aligned} \frac{\alpha^8}{2} \quad & \leq \quad \|\Delta(f, k)\|_{U^2}^4 = \frac{1}{N^4} \sum_r |\widehat{\Delta(f, k)}(r)|^4 \leq \left(\max_r |\widehat{\Delta(f, k)}(r)|^2\right) \frac{1}{N^4} \sum_r |\widehat{\Delta(f, k)}(r)|^2 \\ & = \quad \left(\max_r |\widehat{\Delta(f, k)}(r)|^2\right) \frac{1}{N^3} \sum_n |\Delta(f, k)(n)|^2 \leq \left(\max_r |\widehat{\Delta(f, k)}(r)|^2\right) \frac{1}{N^2} \end{aligned}$$

Therefore, for each $k \in B$, we may find a $\phi(k)$ such that $|\Delta(\widehat{f, k)(\phi}(k))| \geq N\alpha^4/2$.

Proposition

Define f, B, ϕ as above. Then there are at least $\alpha^{64}N^3/2^{12}$ quadruples $(b_1, b_2, b_3, b_4) \in B^4$ such that $b_1 + b_2 = b_3 + b_4$ and $\phi(b_1) + \phi(b_2) = \phi(b_3) + \phi(b_4)$

Proof)

Let $h_u(k) = 1_B(k)e(-u\phi(k)/N)$. We want a lower bound of

$$\frac{1}{N^2}\sum_u\sum_r |\hat{h_u}(-r)|^4 = \sum_{a,b,c,d} 1_B(a)1_B(b)1_B(c)1_B(d)\left(\frac{1}{N}\sum_u e((-\phi(a)-\phi(b)+\phi(c)+\phi(d))u/N)\right)\left(\frac{1}{N}\sum_r e((-a-b+c+d)r/N)\right)$$

$$\begin{aligned}
(N\alpha^4/2)^2\cdot\alpha^8 N/2 &\le \sum_{k\in B} |\widehat{\Delta(f,k)}(\phi(k))|^2 \\
&= \sum_{k\in B}\sum_{x,y}\overline{\Delta(f,k)(x)e(-x\phi(k)/N)}\Delta(f,k)(y)e(-y\phi(k)/N) \\
&= \sum_{k\in B}\sum_{x,y}\overline{f(x)}f(x+k)f(y)\overline{f(y+k)}e((x-y)\phi(k)/N) \\
&= \sum_{x,u}\overline{f(x)}f(x+u)\sum_{k\in B} f(x+k)\overline{f(x+u+k)}e(-u\phi(k)/N) \\
&\le \sum_{x,u}\left|\sum_k \Delta(f,u)(x+k)h_u(k)\right| \\
&\le \left(\sum_{x,u}|1|^2\right)^{1/2}\left(\sum_{x,u}\left|\sum_k \Delta(f,u)(x+k)h_u(k)\right|^2\right)^{1/2} \\
&= N\left(\sum_{x,u}\left|\sum_k \Delta(f,u)(x+k)h_u(k)\right|^2\right)^{1/2} \\
&= N\left(\sum_u\frac{1}{N}\sum_r\left|\hat{F_u}(r)\right|^2\right)^{1/2} \\
&= \sqrt{N}\left(\sum_u\sum_r\left|\sum_x F_u(x)e(-xr/N)\right|^2\right)^{1/2} \\
&= \sqrt{N}\left(\sum_u\sum_r\left|\sum_x\sum_k \Delta(f,u)(x+k)h_u(k)e(-xr/N)\right|^2\right)^{1/2} \\
&= \sqrt{N}\left(\sum_u\sum_r\left|\sum_x\sum_k \Delta(f,u)(x+k)e(-(x+k)r/N)h_u(k)e(kr/N)\right|^2\right)^{1/2} \\
&= \sqrt{N}\left(\sum_u\sum_r\left|\widehat{\Delta(f,u)}(r)\hat{h_u}(-r)\right|^2\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sqrt{N}\left(\sum_u \left(\sum_r \left|\widehat{\Delta(f,u)}(r)\right|^4\right)^{1/2} \left(\sum_r \left|\hat{h_u}(-r)\right|^4\right)^{1/2}\right)^{1/2} \\
&\leq \sqrt{N}\left(\sum_u \left(\sum_r \left|\widehat{\Delta(f,u)}(r)\right|^2\right) \left(\sum_r \left|\hat{h_u}(-r)\right|^4\right)^{1/2}\right)^{1/2} \\
&= \sqrt{N}\left(\sum_u \left(N\sum_n |\Delta(f,u)(n)|^2\right) \left(\sum_r \left|\hat{h_u}(-r)\right|^4\right)^{1/2}\right)^{1/2} \\
&\leq \sqrt{N}\left(\sum_u N^2 \left(\sum_r \left|\hat{h_u}(-r)\right|^4\right)^{1/2}\right)^{1/2} \\
&= N\sqrt{N}\left(\sum_u \left(\sum_r \left|\hat{h_u}(-r)\right|^4\right)^{1/2}\right)^{1/2} \\
&\leq N\sqrt{N}\left(\left(\sum_u 1^2\right)^{1/2} \left(\sum_u\sum_r \left|\hat{h_u}(-r)\right|^4\right)^{1/2}\right)^{1/2} \\
&= N\sqrt{N}\cdot N^{1/4}\left(\sum_u\sum_r \left|\hat{h_u}(-r)\right|^4\right)^{1/4}
\end{aligned}
$$

where $F_u(x) = \sum_k \Delta(f,u)(x+k)h_u(k)$. $\square$

4. Balog-Szemerédi-Gowers Theorem

Balog-Szemerédi-Gowers Theorem says: if A and B have many additive quadruples, than there exist large subsets of A and B such that the sumset is small.

Balog-Szemerédi-Gowers Theorem (version A)

Let A and B be subsets of an abelian group, with $|A| = |B|$. Suppose there are at least $\alpha|A|^3$ additive quadruples $(a_1, b_1, a_2, b_2) \in A \times B \times A \times B$ with $a_1 + b_1 = a_2 + b_2$. Then there are subsets A' of A and B' of B with

$$|A'| \geq \alpha^2|A|/(16\sqrt{2}), \quad |B'| \geq \alpha^2|B|/16, \quad \text{and} \quad |A' + B'| \leq 2^{28}\alpha^{-13}|A|.$$

Balog-Szemerédi-Gowers Theorem (version B)

Let A and B be subsets of an abelian group, with $|A| = |B|$. Let G be a subgraph of the complete bipartite graph between A and B, with G having at least $|A||B|/K$ edges. Suppose that $A +_G B = \{a + b : (a, b) \in G\}$ has cardinality $|A +_G B| \leq L|A|$. Then there are subsets A' of A and B' of B with

$$|A'| \geq |A|/(4\sqrt{2}K), \quad |B'| \geq |B|/(4K), \quad \text{and} \quad |A' + B'| \leq 2^{15}K^5L^3|A|.$$

Version B implies version A

Suppose that we have A and B with $\sum_n r_{A+B}(n)^2 \geq \alpha|A|^3$, where $r_{A+B}(n)$ is the number of ways of writing n as $a+b$ with $a \in A, B \in B$. Then, there are at least $\alpha|A|/2$ popular sum with $r_{A+B}(n) \geq \alpha|A|/2$. Define graph G by letting (a,b) an edge if and only if $a+b$ is a popular sum. There are at least $\alpha|A|/2 \cdot \alpha|A|/2$ edges and there are at most $2|A|/\alpha$ popular sums, which is equal to $|a +_G B|$. Let $K = 4/\alpha^2$ and $L = 2/\alpha$ then version A follows from version B.

Lemma 1

Let G be an undirected bipartite graph having two vertex sets A and B. Suppose that the edge set has cardinality $|A||B|/K$ for some $K \geq 1$. Given $\epsilon \in (0,1)$, there exists a subset A' of A with $|A'| \geq |A|/(\sqrt{2}K)$ such that for at least a proportion $(1-\epsilon)$ of the pairs $(a_1, a_2) \in A' \times A'$, we have at least $\epsilon|B|/(2K^2)$ paths of length 2 in G connecting a_1 and a_2.

Proof)

For $a \in A$, let $B(a)$ denote the set of vertices that are connected to a, and similarly define $A(b)$ for $b \in B$. Let Ω be the subset of $A \times A$ consisting of pairs (a_2, a_2) for which there exists fewer than $\epsilon|B|/(2K^2)$ elements in $B(a_1) \cap B(a_2)$. We have

$$\sum_{b \in B} |A(b)|^2 \geq \frac{(\sum_{b \in B} |A(b)|)^2}{\sum_{b \in B} 1^2} = |A|^2|B|/K^2$$

by Cauchy-Schwarz inequality, and

$$\sum_{b \in B} |A(b)^2 \cap \Omega| = \sum_{b \in B} \sum_{\substack{a_1, a_2 \in A(b) \\ (a_1, a_2) \in \Omega}} 1 = \sum_{(a_1, a_2) \in \Omega} \sum_{b \in B(a_1) \cap B(a_2)} 1 \leq |\Omega|\epsilon|B|/(2K^2) \leq \epsilon|A|^2|B|/(2K^2)$$

Combining the two inequalities, we get

$$\sum_{b \in B} (|A(b)|^2 - \frac{1}{\epsilon}|A(b)^2 \cap \Omega|) \geq \frac{|A|^2|B|}{2K^2}$$

Theref ore, there exists $b \in B$ such that $(|A(b)|^2 - \frac{1}{\epsilon}|A(b)^2 \cap \Omega|) \geq \frac{|A|^2}{2K^2}$. Take $A' = A(b)$. Then $|A'| \geq |A|/(\sqrt{2}K)$ and $|A'^2 \cap \Omega| < \epsilon|A'|^2$, so we are done. $\square$

Lemma 2

Let G be a bipartite graph as above, having an edge set of size $|A||B|/K$. We may extract a set $A' \geq |A|/(4\sqrt{2}K)$, each vertex in A' has degree at least $|B|/(2K)$, and for each $a_1 \in A'$ there exists at least $(1 - 1/(16K))|A'|$ vertices $a_2 \in A'$ such that a_1 and a_2 are joined by at least $|B|/(256K^3)$ paths of length 2.

Proof)

Let G_1 be a subgraph of G formed by removing all vertices with degree $\leq |B|/(2K)$ from A. Let A_1 denote the remaining vertices from A. We remove at most $|A| \cdot |B|/(2K)$ edges from G, so G_1 has at least $|A||B|/(2K)$ edges. Since $a \in A_1$ has degree at most $|B|$, $|A_1| \geq |A|/(2K)$. $K_1 = \frac{|A_1||B|}{\text{number of edges in } G_1} \leq \frac{2K|A_1|}{|A|} \leq 2K$

Let's use lemma 1 with $\epsilon = 1/(32K)$. So there exists $A_2 \subseteq A_1$ such that $|A_2| \geq |A_1|/(\sqrt{2}K_1) \geq |A|/(2\sqrt{2}K)$ such that at least a proportion $(1 - 1/(32K))$ of the pairs $(a_1, a_2) \in A_2 \times A_2$, we have $\geq |B|/(64KK_1^2) \geq |B|/(256K^3)$ paths of length 2 in G_1 connecting a_1 and a_2. We say that such a_1 and a_2 are friendly. So $1/(32K)$ of the pairs are not friendly.

Let A_3 be the set of $a \in A_2$ such that at most a proportion $1/(16K)$ of A_2 are not friendly with a. It is easy to see that $|A_3| > |A_2|/2$ by considering $A_2 \backslash A_3$. Take $A' = A_3$. Then $|A'| > |A_2|/2 \geq |A|/(4\sqrt{2}K)$ and $a \in A_3$ is friendly with at least $1 - 1/(16K)$ of A_3. $\square$

Lemma 3

Let G be a bipartite graph as above having an edge set of size $|A||B|/K$. We may find subsets A' and B' of A and B with $|A'| \geq |A|/(4\sqrt{2}K)$ and $|B'| \geq |B|/(4K)$ such that for any $a \in A'$ and $b \in B'$ there exist $\geq |A||B|/(2^{15}K^5)$ paths of length three joining a and b.

Proof)

Take A' to be the A' obtained from lemma 2. Take B' to be the set of vertices adjacent to at least $|A'|/(8K)$ elements from A'. There are at least $|A'| \cdot |B|/(2K)$ edges coming out of $|A'|$ so $|B'|$ is at least $|B|/(4K)$.

For any $a \in A'$ and $b \in B'$, a have at least $(1 - 1/(16K))|A'|$ friendly elements in A'. B is adjacent to at least $|A'|/(8K)$ elements from A'. Therefore, there are at least $|A'|/(16K)$ elements in A' that are friendly with a and adjacent to b. Each such element has at least $|B|/(256K^3)$ paths of length 2 to a, so there are at least $|A'|/(16K) \cdot |B|/(256K^3) \geq |A||B|/(2^{15}K^5)$ paths of length 3 from a to b. $\square$

Proof of Balog-Szemerédi-Gowers Theorem

Take A' and B' as in lemma 3. So given $a \in A'$ and $b \in B'$, we can find at least $|A||B|/(2^{15}K^5)$ pairs of $a' \in A'$ and $b' \in B'$ such that $(a, b') = x$, $(b', a') = y$, and $(a', b) = z$ are all edges in G. In other words, $a + b'$, $b' + a'$, and $a' + b$ are all in $A +_G B$.

Now note that $a + b = (a + b') - (a' + b') + (a' + b) = x - y + z$. We have at least $|A||B|/(2^{15}K^5)$ solutions to $a + b = x - y + z$ where $x, y, z \in A +_G B$. However, there are only $|A +_G B|^3 \leq L^3|A|^3$ many possible choices for x, y, z. Therefore, there are at most

$$\frac{L^3|A|^3}{|A||B|/(2^{15}K^5)} = 2^{15}K^5L^3|A|$$

distinct possibility for $a + b$. $\square$

References

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