PICARD GROUP OF ALGEBRAIC CURVES

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1. Preliminaries

We begin by briefly defining some objects we will be dealing with.

1.1. Divisor Classes.

Definition 1.1. A **divisor** of an algebraic curve, X, is a function $D : X \to \mathbb{Z}$ whose support is discrete. This is often denoted as a formal sum $D = \sum_{p \in X} D(p) \cdot p$. We can define addition of two divisors in a natural way, and call the additive group (this is a free abelian group) as $\mathrm{Div}(X)$.

Definition 1.2. The **degree** of a divisor D is $\sum_{p \in X} D(p)$.

Definition 1.3. Given $f \in \mathcal{M}^*(X)$, we say $div(f) = \sum_{p \in X} ord_p(f) \cdot p$ is the associated divisor of f. Any divisors of this form is called **principal divisor**.

Lemma 1.4. $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$, $\mathrm{div}(f/g) = \mathrm{div}(f) - \mathrm{div}(g)$

Definition 1.5. We call the additive group of principal divisors PDiv(X). Define the **divisor class group** as $\mathrm{Cl}(X) := \mathrm{Div}(X)/\,\mathrm{PDiv}(X)$.

1.2. Line Bundles.

Definition 1.6. Line bundle consists of

(1) Total space L
(2) Base space X
(3) Continuous projection $\pi : L \to X$

such that

(1) $\pi^{-1}(p) \cong \mathbb{C}$ for any $p \in X$. ($p \in X$ maps back to something homeomorphic to $\mathbb{C}$ by π)
(2) There exists an open cover of X, $\{U_i\}_{i \in I}$ and homeomorphisms $\phi_i : \pi^{-1}(U_i) \to \mathbb{C} \times U_i$ such that $pr_2 \circ \phi_i = \pi$ on $\pi^{-1}(U_i)$. (ϕ_i projects to π on U_i)
(3) $\phi_j \circ \phi_i^{-1} : \mathbb{C} \times (U_i \cap U_j) \to \mathbb{C} \times (U_i \cap U_j)$ has the form $(v, p) \mapsto (t_{i,j} \cdot v, p)$ for some $t_{i,j} \in \mathcal{O}^*(U_i \cap U_j)$. (charts are compatible)

If two altases have charts that are compatible with each other, we say that they define the same line bundle.

Definition 1.7. Let $\pi_1 : L_1 \to X$ and $\pi_2 : L_2 \to X$ be two line bundles. $\alpha : L_1 \to L_2$ is a **line bundle homomorphism** if $\pi_2 \circ \alpha = \pi_1$ and for every pair of charts $\phi_1 : \pi_1^{-1}(U_1) \to \mathbb{C} \times U_1$ and $\phi_2 : \pi_2^{-1}(U_2) \to \mathbb{C} \times U_2$, the composition $\phi_2 \circ \alpha \circ \phi_1^{-1}$ has the form $(s, p) \mapsto (f(p)s, p)$ for some regular function $f \in \mathcal{O}(U_1 \cap U_2)$.

Definition 1.8. A line bundle homomorphism is a **line bundle isomorphism** if there exists an inverse homomorphism. Let LB(X) be the set of isomorphic classes of line bundles on X.

1.3. **First Cohomology Group.**

Definition 1.9. Coboundary operator

(1) $\delta^0 : C^0(U, \mathcal{F}) \to C^1(U, \mathcal{F})$ maps (f_i) to $(g_{i,j})$ where $g_{i,j} = f_j - f_i$.
(2) $\delta^1 : C^1(U, \mathcal{F}) \to C^2(U, \mathcal{F})$ maps $(f_{i,j})$ to $(g_{i,j,k})$ where $g_{i,j,k} = f_{j,k} - f_{i,k} + f_{i,j}$.

Definition 1.10. First Cohomology Group

$$\check{H}^1(U, \mathcal{F}) = \frac{\ker \delta^1}{\text{im } \delta^0}$$

This contains classes of cocycles $(f_{i,k} = f_{i,j} + f_{j,k})$

1.4. **Zariski Topology.**

Definition 1.11. The **Zariski topology** on an algebraic curve, X, is the topology whose open sets are empty set and X with finite set of points removed.

Remark 1.12.

- The Zariski topology is not Hausdorff.
- It is compact.
- Any two nonempty open sets intersect.
- It is a subtopology of the usual topology.
- Sheaf on X with usual topology induces sheaf on X_{Zar}

1.5. **Hopf Manifold.**

Definition 1.13. Let $\mu_i \in \mathbb{C}$ for $i = 1, 2, \cdots, n$ such that $0 < |\mu_i| < 1$. On $\mathbb{C}^n \backslash 0$, define an equivalence relation such that $(x_1, \cdots, x_n) \sim (y_1, \cdots, y_n)$ if and only if $y_i = \mu_i^m x_i$ for all i for some $m \in \mathbb{Z}$. We define n-dimensional **Hopf manifold** to be $X = (\mathbb{C}^n \backslash 0)/\sim$.

Equivalently, let μ be an action on $\mathbb{C}^n \backslash 0$ mapping $(z_1, \cdots, z_n)$ to $(\mu_1 z_1, \cdots, \mu_n z_n)$. Define X to be the quotient $(\mathbb{C}^n \backslash 0)/\langle \mu \rangle$.

Remark 1.14. 1-dimensional Hopf manifold is a torus, thus an elliptic curve. We can see this in two ways

(1) $(\mathbb{C} \backslash 0)$ is homeomorphic to $\mathbb{C}/\mathbb{Z}$. Equivalence classes are isomorphic to $\mathbb{Z}$, so $X \cong \mathbb{C}/\mathbb{Z}^2$.
(2) For each equivalence class, we can pick a representative z such that $|\mu_1| < |z| \leq 1$. So X can be seen as an annulus, with the two circular boundaries identified to each other, which yields us a torus.

We denote 1-dimensional Hopf manifold X by $T(\mu_1)$.

Remark 1.15. 2-dimensional Hopf manifold with $\mu_1 = \mu_2$ is isomorphic to $\mathbb{P}^1 \times T$ where T is a torus. $\langle \mu \rangle$ is a subgroup of $L = \{(x, x) : x \in \mathbb{C}^*\} \cong \mathbb{C}^*$ where (x, x) acts on $\mathbb{C}^2 \backslash 0$ mapping (z, w) to (xz, xw), so

$$X = (\mathbb{C}^2 \backslash 0)/\langle \mu \rangle = ((\mathbb{C}^2 \backslash 0)/L) \times (L/\langle \mu \rangle) \cong ((\mathbb{C}^2 \backslash 0)/\mathbb{C}^*) \times (\mathbb{C}^*/\langle \mu \rangle) \cong \mathbb{P}^1 \times T$$

It can also be seen as a fibre bundle on $\mathbb{P}^1$ with fibre T. Given any $(z, w) \in \mathbb{C}^2 \backslash 0$, (z, w) is on an unique complex line that goes through 0 and this is represented by $[z : w]$. Each equivalence class lie on a single line. On each line, μ acts on it as usual, so we get $\mathbb{C}^*/\mathbb{Z} \cong T$.

2. Picard Group

We will see that for an algebraic curve X, the divisor classes of X is equal to the group of line bundles on X and a first cohomology group, $\hat{H}^1(X_{Zar}, \mathcal{O}^*)$. This group is called Picard group, $\mathrm{Pic}(X)$.

2.1. Divisor Classes and First Cohomology Group.

Consider the map $\mathrm{div} : \mathcal{M}^*_{X,alg} \to \mathrm{Div}_{X,alg}$, where $\mathcal{M}^*_{X,alg}$ is the sheaf of meromorphic functions on X that is not identical to zero, and $\mathrm{Div}_{X,alg}(U)$ is the divisors supported by U. This is an onto map. The kernel of this map is $\mathcal{O}^*_{X,alg}$. So we get a short exact sequence:

$$0 \to \mathcal{O}^*_{X,alg} \to \mathcal{M}^*_{X,alg} \to \mathrm{Div}_{X,alg} \to 0$$

which induces a long exact sequence:

$$\begin{aligned} 0 \quad \to \quad & \check{H}^0(\mathcal{O}^*_{X,alg}) \to \check{H}^0(\mathcal{M}^*_{X,alg}) \to \check{H}^0(\mathrm{Div}_{X,alg}) \\ \to \quad & \check{H}^1(\mathcal{O}^*_{X,alg}) \to \check{H}^1(\mathcal{M}^*_{X,alg}) \to \check{H}^1(\mathrm{Div}_{X,alg}) \to \cdots \end{aligned}$$

$\check{H}^0(\mathcal{O}^*_{X,alg})$ is $\mathcal{O}^*_{X,alg}(X)$ which is holomorphic functions with no poles and zeros, so it is isomorphic to $\mathbb{C}^*$. $\check{H}^0(\mathcal{M}^*_{X,alg}) = \mathcal{M}^*_{X.alg}(X)$ is just simply $\mathcal{M}(X)\backslash\{0\}$. $\check{H}^0(\mathrm{Div}_{X,alg})$ is $\mathrm{Div}(X)$. Finally $\check{H}^1(\mathcal{M}^*_{X,alg})$ is 0 by the following proposition. ($\mathcal{M}^*_{X,alg}$ is a constant sheave because we are in Zariski topology and every two open sets intersect)

Proposition 2.1. $\check{H}^1(X_{Zar}, \underline{G}) = 0$

Proof. Because X_{Zar} is compact, we can take a finite open covering $\{U_i\}_{1\leq i\leq n}$. $f_{i,j} = -f_{j,i}$ and $f_{i,k} = f_{i,j} + f_{j,k}$ so cocycles are determined from $f_{i,i+1}$ (also note that U_i and U_{i+1} always intersect). Take $g_i = \sum_{k=1}^{i-1} f_{k,k+1}$. So any 1-cocycle is a 0-coboundary.

Rewriting the long exact sequence:

$$0 \to \mathbb{C}^* \to \mathcal{M}(X)\backslash\{0\} \stackrel{div}{\to} \mathrm{Div}(X) \to \check{H}^1(\mathcal{O}^*_{X,alg}) \to 0$$

Image of div map is $\mathrm{PDiv}(X)$, so $\mathrm{Cl}(X) = \mathrm{Div}(X)/\,\mathrm{PDiv}(X) = \check{H}^1(\mathcal{O}^*_{X,alg})$.

2.2. Line Bundles and First Cohomology Group.

We will prove that $LB(X) \stackrel{H_L}{\cong} \check{H}^1(\mathcal{O}^*_{X,alg})$ in three steps.

Proposition 2.2. Transition functions yield a well-defined cocycle class in $\check{H}^1$. (i.e. H_L is a function)

Proof. Given a line bundle, fix an altas with open covering $\mathcal{U} = \{U_i\}$ with transition functions $t_{i,j}$. The transition functions satisfy the cocycle conditions, so they represent a 1-cocycle class in $\check{H}^1(\mathcal{U}, \mathcal{O}^*)$, which induces a class in $\check{H}^1(X_{Zar}, \mathcal{O}^*)$.

If we use a different chart to get the transition map but with the same covering, say use ϕ'_i instead of ϕ_i, then because of the compatibility, we get a nowhere zero regular function s_i. Then $t_{i,j} = (s_i/s_j) \cdot t'_{i,j}$. So the 1-cocycles are off by multiplication by a 1-coboundary. Therefore, they are in the same class in $\check{H}^1(\mathcal{U}, \mathcal{O}^*)$ and $\check{H}^1(X_{Zar}, \mathcal{O}^*)$.

We wish to see that even choosing different open covering does not affect the class. It is sufficient to consider refinement $\mathcal{V}$ of $\mathcal{U}$. Let r be the refining map (i.e. $V_k \subseteq U_{r(k)}$ for all k). Then, we can use the same chart by restricting the chart on $\mathcal{U}$ to $\mathcal{V}$. Then the corresponding cocycles are equal in the limit group $\check{H}^1(X_{Zar}, \mathcal{O}^*)$.

Proposition 2.3. Given transition functions satisfying the cocycle condition, there exists a line bundle with such transition functions. (i.e. H_L is onto)

Proof. $(s,p) \in \mathbb{C} \times U_j$ is identified with $(t_{i,j}s, p) \in \mathbb{C} \times U_i$. Let L be $\coprod_i (\mathbb{C} \times U_i)/\sim$, where $(s,p) \sim (t_{i,j}s, p)$. The natural map $\psi_i : \mathbb{C} \times U_i \to L$ is injective. Let $\phi_i : L_i \to \mathbb{C} \times U_i$ to be the inverse map for some $L_i \subseteq L$. We can check that $t_{i,j}$ is the transition map for ϕ_i and ϕ_j. So we have constructed a line bundle from transition functions.

Proposition 2.4. If 2 line bundle give the same cocycle class then they are isomorphic. (i.e. H_L is 1-1)

Proof. Let $\pi_i : L_i \to X$ have opening covering $\{U_i^{(k)}\}$ supporting atlas $\{\phi_i^{(k)}\}$ for $k = 1, 2$. Say H_L maps both line bundles to the same class in the first cohomology group. By using the common refinement, we can assume that both line bundles use the same open cover $\{U_i\}$. The two cocyles determined from the two charts live in $\check{H}^1(\mathcal{U}, \mathcal{O}^*)$ and is the same in $\check{H}^1(X_{Zar}, \mathcal{O}^*)$, so we can further refine $\mathcal{U}$ so that the cocycles are in the same class in $\check{H}^1(\mathcal{U}, \mathcal{O}^*)$, i.e. only differ by a coboundary.

So there exists nowhere zero regular functions s_i on U_i such that $t_{i,j}^{(1)} \cdot s_i/s_j = t_{i,j}^{(2)}$. For each i, define $S_i : \mathbb{C} \times U_i \to \mathbb{C} \times U_i$ by $S_i(z_i, p) = (s_i z_i, p)$. Consider an alternative charts for L_1, $\phi_i^{(1a)} : \pi^{-1}(U_i) \to \mathbb{C} \times U_i$ by $\phi_i^{(1a)} = S_i \circ \phi_i^{(1)}$. This chart is compatible with charts $\phi_i^{(1)}$. We also get $t_{i,j}^{(1a)} = t_{i,j}^{(1)} \cdot s_i/s_j = t_{i,j}^{(2)}$.

Then let $\alpha : L_1 \to L_2$ be $\alpha = (\phi_i^{(2)})^{-1} \circ \phi_i^{(1a)}$ on $\pi^{-1}(U_i)$. $(\phi_i^{(2)})^{-1} \circ \phi_i^{(1a)} = (\phi_j^{(2)})^{-1} \circ (\phi_j^{(2)} \circ (\phi_i^{(2)})^{-1}) \circ \phi_i^{(1a)} = (\phi_j^{(2)})^{-1} \circ (\phi_j^{(1a)} \circ (\phi_i^{(1a)})^{-1}) \circ \phi_i^{(1a)} = (\phi_j^{(2)})^{-1} \circ \phi_j^{(1a)}$ on $\pi^{-1}(U_i \cap U_j)$, so this is well-defined. $\pi_2 \circ \alpha = \pi_1$ and $\phi_j^{(2)} \circ \alpha \circ (\phi_i^{(1a)})^{-1} = \phi_j^{(2)} \circ ((\phi_j^{(2)})^{-1} \circ \phi_j^{(1a)}) \circ (\phi_i^{(1a)})^{-1} = \phi_j^{(1a)} \circ (\phi_i^{(1a)})^{-1}$. so it has the form $(s,p) \mapsto (t_{i,j}s, p)$ and we are done.

3. Examples

We will now see some examples of Picard groups.

3.1. Projective space.

What is $\mathrm{Pic}(\mathbb{P}^1)$?

(1) Divisor Classes

Consider a divisor $D = p - q$, where $p = [a : b]$ and $q = [c : d]$. D is principal since the divisor of rational function $f = \frac{bx-ay}{dx-cy}$.

Say divisor D has degree 0. Then it can be written as $\sum_i p_i - q_i$. For every i, $p_i - q_i = div(f_i)$ for some rational function f_i. So $D = \mathrm{div}(\prod f_i)$ and D is also principal. Therefore, all divisors with degree 0 is principal.

Every principal divisor has degree 0. So $\mathrm{PDiv}(\mathbb{P}^1) = \mathrm{Div}^0(\mathbb{P}^1)$ and

$$\mathrm{Pic}(\mathbb{P}^1) \cong \mathrm{Cl}(\mathbb{P}^1) = \mathrm{Div}(\mathbb{P}^1)/\,\mathrm{PDiv}(\mathbb{P}^1) = \mathrm{Div}(\mathbb{P}^1)/\,\mathrm{Div}^0(\mathbb{P}^1) \cong \mathbb{Z}$$

(2) Line Bundles

$U_0 = \mathbb{P}^1\backslash 0$ and $U_\infty = \mathbb{P}^1\backslash\infty$ is an open covering. Since both open sets are contractible to a point, the line bundle is trivial when restricted to these open sets. Therefore, any line bundle can be described by the open covering with a transition function.

Let local coordinates be $U_0 = \{[1 : z] : z \in \mathbb{C}\}$ and $U_\infty = \{[w : 1] : w \in \mathbb{C}\}$ with $z = 1/w$ on the intersection. Then the transition map, $t : \mathbb{C} \to \mathbb{C}$, must be nowhere zero regular function on $U_0 \cap U_\infty = \mathbb{P}\backslash\{0, \infty\}$, but may have zeros and poles at 0 and ∞. The possible transition function is $t = z^n$ for $n \in \mathbb{Z}$ and $1/z^n = w^n$.

(3) First Cohomology

$exp : \mathcal{O} \to \mathcal{O}^*$, $f \mapsto e^{2\pi i f}$ is an onto map with kernel $\underline{\mathbb{Z}}$. This induces a short exact sequence $0 \to \underline{\mathbb{Z}} \to \mathcal{O} \to \mathcal{O}^* \to 0$. Therefore,

$$\begin{aligned} 0 \quad &\to \quad \check{H}^0(\underline{\mathbb{Z}}) \to \check{H}^0(\mathcal{O}) \to \check{H}^0(\mathcal{O}^*) \\ &\to \quad \check{H}^1(\underline{\mathbb{Z}}) \to \check{H}^1(\mathcal{O}) \to \check{H}^1(\mathcal{O}^*) \\ &\to \quad \check{H}^2(\underline{\mathbb{Z}}) \to \check{H}^2(\mathcal{O}) \to \check{H}^2(\mathcal{O}^*) \to \cdots \end{aligned}$$

$\check{H}^1(\mathcal{O})$ and $\check{H}^2(\mathcal{O})$ is 0 (using Dolbeault cohomology), so $\mathrm{Pic}(\mathbb{P}^1) \cong \check{H}^1(\mathbb{P}^1, \mathcal{O}^*) \cong \check{H}^2(\mathbb{P}^1, \underline{\mathbb{Z}}) \cong \mathbb{Z}$ (using singular cohomology).

3.2. **Elliptic curve.**

Proposition 3.1. $\mathrm{Pic}(X) \cong \mathbb{Z} \times X$

Proof. Let $p_0 \in X$ be the only singularity of order 3.

(1) The divisor $(p - p_0) + (q - q_0)$ is equivalent to $r - p_0$ for some $r \in X$

Let L_1 be a line passing through p and q. By Bezout's theorem, L_1 passes through another point, say s. Let L_2 be a line passing through s and p_0. L_2 also passes through another point, say r. L_1 and L_2 are both linear and $\mathrm{div}(L_1) = p + q + s$, $\mathrm{div}(L_2) = s + p_0 + r$, so $\mathrm{div}(L_1) - \mathrm{div}(L_2) = p + q - r - p_0 = (p - p_0) + (q - p_0) - (r - p_0)$ is a divisor of a meromorphic function. Therefore, $(p - p_0) + (q - q_0) \equiv r - p_0$.

(2) The divisor $-(p - p_0)$ is equivalent to $q - p_0$ for some $q \in X$

Let L_1 be a line passing through p and p_0. By Bezout's theorem, L_1 passes through another point, say q. Let L_2 be the tangent line at p_0. L_2 has order 3 zero at p_0. L_1 and L_2 are both linear and $\mathrm{div}(L_1) = p + p_0 + q$, $\mathrm{div}(L_2) = 3p_0$, so $\mathrm{div}(L_1) - \mathrm{div}(L_2) = p + q - 2p_0 = (p - p_0) + (q - p_0)$ is a divisor of a meromorphic function. Therefore, $-(p - p_0)) \equiv q - p_0$.

If D is a divisor of degree 0, then

$$D = \sum_i (q_i - r_i) = \sum_i (q_i - p_0) + \sum_i -(r_i - p_i) \equiv \sum_i (q_i - p_0) + \sum_i (r'_i - p_0) = p - p_0$$

for some $p \in X$ by the two lemmas. Therefore, $\mathrm{Pic}(X) \subseteq \mathbb{Z} \times X$.

Suppose that $p - p_0 \equiv q - p_0$ for some $p \neq q$. Then $\mathrm{div}(f) = p - q$ for some meromorphic function f. Then f has the only pole of order 1 at q, which is a contradiction. Therefore, $\mathrm{Pic}(X) \cong \mathbb{Z} \times X$.

Let us look at the Picard group as the group of line bundles instead of the divisor group. Consider a covering $\pi : \mathbb{C}^* \to X = \mathbb{C}^*/\langle\mu\rangle$. Any holomorphic line bundle on $\mathbb{C}^*$ is trivial (read p229 of [3]), so if E is a holomorphic line bundle over X, then $\pi^*(E) \cong \mathbb{C}^* \times \mathbb{C}$. Therefore, $\mathbb{C}^* \times \mathbb{C}/\langle \mu \times A \rangle$ is a line bundle on X for any $A \in \Gamma(\mathbb{C}^*, \mathcal{O}^*)$, where the action is $\mu \times A : \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}$ mapping (z, v) to $(\mu z, A(z)v)$. We represent such line bundle E with the holomorphic function A.

Proposition 3.2. *A and $\tilde{A}$ define the isomorphic line bundle if and only if $\tilde{A}(z) = B(\mu z)A(z)B(z)^{-1}$.*

Proof. Let ξ be a coordinate of the trivial line bundle $\mathbb{C}^* \times \mathbb{C}$. Then, the action $\mu \times A$ maps (z, ξ) to $(\mu z, A(z)\xi)$. Change the coordinate ξ to $\eta := B\xi$ for some $B \in \Gamma(\mathbb{C}^*, \mathcal{O}^*)$.

$$\begin{aligned} (z,\xi) \mapsto (\mu z, A(z)\xi) \quad &\Rightarrow \quad (z, B(z)\xi) \mapsto (\mu z, B(\mu z)A(z)\xi) \\ &\Rightarrow \quad (z, \eta) \mapsto (\mu z, B(\mu z)A(z)B(z)^{-1}\eta) \end{aligned}$$

Let $\tilde{A}$ be $B(\mu z)A(z)B(z)^{-1}$. Then, the action $\mu \times \tilde{A}$ maps (z, η) to $(\mu z, \tilde{A}(z)\eta)$. So we have two actions, both doing the same thing but for different coordinates. Therefore, $\mathbb{C}^* \times \mathbb{C}/\langle \mu \times A \rangle \cong \mathbb{C}^* \times \mathbb{C}/\langle \mu \times \tilde{A} \rangle$. This argument also works the other direction, so we are done.

Proposition 3.3. *Isomorphism class of such line bundles form a group $\mathbb{Z} \times T(\mu)$*

Proof. Let $A \in \mathcal{O}^*(\mathbb{C}^*)$ represent of a line bundle. Then $A(z) = z^d e^{a(z)}$ for some $a(z) \in \mathcal{O}(\mathbb{C}^*)$ and $d \in \mathbb{Z}$. Similarly, $B(z) = z^l e^{b(z)}$. Then,

$$\begin{aligned} \tilde{A}(z) &= B(\mu z)A(z)B(z)^{-1} \\ &= ((\mu z)^l e^{b(\mu z)})(z^d e^{a(z)})(z^{-l} e^{-b(z)}) \\ &= \mu^l z^d e^{b(\mu z)+a(z)-b(z)} \\ &= \mu^l z^d e^{\sum_{i\in\mathbb{Z}}(b_i\mu^i + a_i - b_i)z^i} \end{aligned}$$

Choose $b(z)$ such that $b_i = a_i/(1-\mu^i)$ for $i \in \mathbb{Z}\backslash 0$ and $|\mu| < \alpha \leq 1$ where $\alpha := e^{a_0}\mu^l$. Then $\tilde{A}(z) = \alpha z^d$.

Therefore, we see that all line bundles must be of the form $\mathbb{C}^* \times \mathbb{C}/\langle \mu \times A \rangle$.

3.3. Hopf Manifolds.

It is known that $\mathrm{Pic}(X) \cong \mathbb{C}^*$ for any n-dimensional Hopf Manifold where $n \geq 2$ (read p226 of [4]).

Similarly as the elliptic curve case, $(\mathbb{C}^2\backslash 0) \times \mathbb{C}/\langle A \times \mu \rangle$ are line bundles on $X = (\mathbb{C}^2\backslash 0)/\langle \mu \rangle$, where $A \in \Gamma(\mathbb{C}^2\backslash 0, \mathcal{O}^*)$ and A and $\tilde{A}$ define the isomorphic line bundle if and only if $\tilde{A}(z) = B(\mu z)A(z)B(z)^{-1}$.

Proposition 3.4. *Isomorphism class of such line bundles form a group $\mathbb{C}^*$*

Proof. Let $A \in \Gamma(\mathbb{C}^2\backslash 0, \mathcal{O}^*)$ represent of a line bundle. Then $A(z) = e^{a(z,w)}$ for some $a(z) \in \mathcal{O}(\mathbb{C}^2\backslash 0)$. Similarly, $B(z) = e^{b(z,w)}$. Then,

$$\begin{aligned}\tilde{A}(z,w) &= B(\mu_1 z, \mu_2 w)A(z,w)B(z,w)^{-1} \\ &= e^{b(\mu_1 z, \mu_2 w)+a(z,w)-b(z,w)} \\ &= e^{\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}(b_{ij}\mu_1^i\mu_2^j+a_{ij}-b_{ij})z^i w^j}\end{aligned}$$

Choose $b(z,w)$ such that $b_{ij} = a_{ij}/(1-\mu_1^i\mu_2^j)$ for $(i,j) \in \mathbb{Z}^2\backslash 0$. Thus $\tilde{A}(z) = e^{a_{00}}$.

Therefore, we see that all line bundles must be of the form $\mathbb{C}^* \times \mathbb{C}/\langle\mu\times A\rangle$. Note that this proposition works for any n-dimensional Hopf manifold where $n \geq 2$.

We saw that $X \cong \mathbb{P}^1 \times T$ if $\mu_1 = \mu_2$. If we take a subset $X_1 = \{(z,w) \in \mathbb{C}^2 : w \neq 0\} \subseteq X = \mathbb{C}^2\backslash 0$ which corresponds to $U_1 \times T \subseteq \mathbb{P}^1 \times T$ where $U_1 = \mathbb{P}^1\backslash\infty \cong \mathbb{C}$, the Picard group is $\mathrm{Pic}(X_1) \cong \mathrm{Pic}(U_1 \times T) \cong \mathrm{Pic}(T) \cong \mathbb{Z} \times T$.

The same Picard group can be calculated by finding the factor of automorphy. We can do the same calculation we did in the previous proposition, but now we could have a zero or pole on $w = 0$ axis. So instead of $A(z) = e^{a(z,w)}$, we get more general form $A(z) = w^d e^{a(z,w)}$ for some $d \in \mathbb{Z}$. Following through the calculation, we can confirm that $\mathrm{Pic}(X_1)$ is indeed $\mathbb{Z}\times T$. This calculation also works if $\mu_1 \neq \mu_2$.

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