Littlewood-Richardson Numbers

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Definition

A partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_d)$ is a decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$.
We say that the size of λ is $|\lambda| = \sum \lambda_i$.
We write $\lambda \vdash m$ if $|\lambda| = m$.

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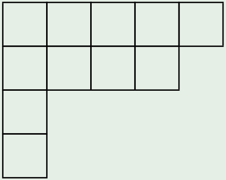
Example

There are 5 partitions of 4: (4), (3,1), (2,2), (2,1,1), (1,1,1,1).

Example

Young diagram of $(5, 4, 1, 1)$ is



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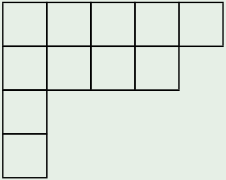
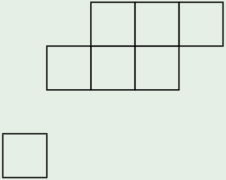


Example

Skew Young diagram of $(5, 4, 1, 1)/(2, 1, 1)$ is



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A semistandard Young tableau of $(5, 4, 1, 1)/(2, 1, 1)$ is

		1	1	3
	3	4	5	
1				

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Example

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	3	4	5	
1				

The content of this tableau is $(3,0,2,1,1)$.

Definition

A Schur function s_λ is

$$\sum x_1^{c_1(T)} x_2^{c_2(T)} \dots$$

where the sum is over all young tableux T of shape λ, and $c_i(T)$ denote the number of appearance of i in T.

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$s_{\square} = x_1 + x_2 + \cdots$
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Theorem

s_λ is symmetric and $\{s_\lambda | \lambda \vdash m\}$ is a basis of the vector space of symmetric functions of degree m.

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$$
\begin{aligned}
(s_{\square})^2 &= (x_1 + x_2 + \cdots)^2 \\
&= x_1^2 + x_2^2 + x_3^2 + \cdots + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 + \cdots \\
&= (x_1^2 + x_2^2 + x_3^2 + \cdots + x_1x_2 + x_1x_3 + x_2x_3 + \cdots) \\
&\quad +(x_1x_2 + x_1x_3 + x_2x_3 + \cdots) \\
&= s_{\square\square} + s_{\begin{array}{c}\square\\[-4pt]\square\end{array}}
\end{aligned}
$$

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A Littlewood-Richarson Tableau is a skew semistandard Young tableau T if there exists a Young tableau U of a partition such that the number of i's in row j of T is equal to the number of j's in row i of U.

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Example

T=

					1	1	1
					2	2	
				2	3		
			1				
		1	2				
1	1	3					

, U=

1	1	1	4	5	6	6
2	2	3	5			
3	6					

Theorem

The Littlewood-Richardson Number $c^{\lambda}_{\mu\nu}$ is equal to the number of Littlewood-Richarson tableaux of shape λ/μ with content ν.

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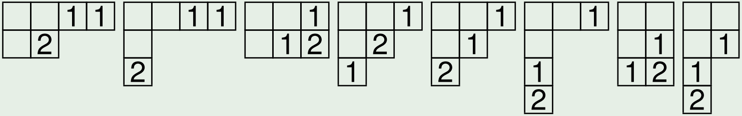
Example

$s_{(2,1)}s_{(2,1)} =$

$s_{(4,2)} + s_{(4,1,1)} + s_{(3,3)} + 2 \cdot s_{(3,2,1)} + s_{(3,1,1,1)} + s_{(2,2,2)} + s_{(2,2,1,1)}$



Question

When is $c_{\mu\nu}^{\lambda} = 0$?

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Theorem

$c^{\lambda}_{\mu\nu} \neq 0$ if and only if there exists $A \in GL(\mathbb{C}, d)$ and $B \in GL(\mathbb{C}, n-d)$ such that $\tau_{\mu} \oplus \tau_{\lambda}^{\vee} \oplus A\tau_{\nu}B = M_{d,n-d}(\mathbb{C})$.

Example

$c_{\square,\square}^{\square\square} \neq 0$ since

$$\tau_{\square} = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\rangle,$$

$$\tau_{\square\square}^{\vee} = \left\langle \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle,$$

and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tau_{\square} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \left\langle \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\rangle$,

We can check the right hand side by calculating whether the vectors $Ae_{i,j}B$ ((i,j) are in ν) are linearly independent on $\tau_{\lambda/\mu}$.

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Example

$c^{(4,3,2)}_{(2,2),(2,2,1)} \neq 0$

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$c^{(4,3,2)}_{(2,2),(2,2,1)} \neq 0$

LR tableau =

		1	1
		2	
2	3		

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Example

$c^{(4,3,2)}_{(2,2),(2,2,1)} \neq 0$

$$\text{LR tableau} = \begin{array}{cccc} & & 1 & 1 \\ & & 2 & \\ 2 & 3 & & \end{array}$$

$$\text{matrix} = \begin{bmatrix} A_{11}B_{13} & A_{11}B_{23} & A_{12}B_{13} & A_{12}B_{23} & A_{13}B_{13} \\ A_{11}B_{14} & A_{11}B_{24} & A_{12}B_{14} & A_{12}B_{24} & A_{13}B_{24} \\ A_{21}B_{13} & A_{21}B_{23} & A_{22}B_{13} & A_{22}B_{23} & A_{23}B_{13} \\ A_{31}B_{11} & A_{31}B_{21} & A_{32}B_{11} & A_{32}B_{21} & A_{33}B_{11} \\ A_{31}B_{12} & A_{31}B_{22} & A_{32}B_{12} & A_{32}B_{22} & A_{33}B_{12} \end{bmatrix}$$

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Example

$c^{\yng(4,3,2)}_{\yng(2,2),\yng(2,2,1)} \neq 0$

LR tableau = $\begin{array}{cccc} & & \boxed{1} & \boxed{1} \\ & & \boxed{2} & \\ \boxed{2} & \boxed{3} & & \end{array}$

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$$\text{determinant} = A_{(123)(123)}A_{(13)(12)}B_{(12)(34)}B_{(12)(12)}B_{(1)(3)}$$