Cyclic Sieving Phenomenon

David Rhee

University of Waterloo

February 10, 2011

Definition

- X = finite set

Definition

- X = finite set
- $C = \{1, c, c^2, \cdots, c^{n-1}\}$, a finite cyclic group acting on X

Definition

- X = finite set
- $C = \{1, c, c^2, \cdots, c^{n-1}\}$, a finite cyclic group acting on X
- $\zeta = e^{\frac{2\pi i}{n}} \in \mathbb{C}$, the root of unity of order n

Definition

- X = finite set
- $C = \{1, c, c^2, \cdots, c^{n-1}\}$, a finite cyclic group acting on X
- $\zeta = e^{\frac{2\pi i}{n}} \in \mathbb{C}$, the root of unity of order n
- $f(q)$ = a polynomial with rational coefficients

Definition

- X = finite set
- $C = \{1, c, c^2, \cdots, c^{n-1}\}$, a finite cyclic group acting on X
- $\zeta = e^{\frac{2\pi i}{n}} \in \mathbb{C}$, the root of unity of order n
- $f(q)$ = a polynomial with rational coefficients

We say that the triple $(X, C, f(q))$ exhibits the cyclic sieving phenomenon (CSP) if for any nonnegative integer d, we have that the fixed point set cardinality $|X^{c^d}|$ is equal to the polynomial evaluation $f(\zeta^d)$.

Remark

$f(1)$ is equal to the number of elements in X.

Remark

$f(1)$ is equal to the number of elements in X.

Remark

$f(q)$ is unique up to the cyclotomic polynomial $\Phi_n(q)$.

Remark

$f(1)$ is equal to the number of elements in X.

Remark

$f(q)$ is unique up to the cyclotomic polynomial $\Phi_n(q)$.

Remark

If $f(q) = \sum_{k=0}^{n-1} a_k q^k$ where a_k is the number of C-orbits in X with stabilizer order dividing k, then $(X, C, f(q))$ exhibits the CSP.

Definition

- $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$

Definition

- $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$
- $[n]_q! = [1]_q[2]_q \cdots [n]_q$

Definition

- $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$
- $[n]_q! = [1]_q[2]_q \cdots [n]_q$
- $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$

Definition

- $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$
- $[n]_q! = [1]_q[2]_q \cdots [n]_q$
- $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$

These q-analogs are all polynomials in q and are our ordinary numbers, factorials, and binomial coefficients as q approaches 1.

Definition

- $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$
- $[n]_q! = [1]_q[2]_q \cdots [n]_q$
- $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$

These q-analogs are all polynomials in q and are our ordinary numbers, factorials, and binomial coefficients as q approaches 1.

When $(X, C, f(q))$ exhibits the CSP, f turns out to be the q-analog of the number of elements in X in many cases.

Let X be the set of k-multisets of $[n]$.
Let C be a cyclic subgroup of S_n that is generated by the cycle $c = (1, 2, 3, \cdots, n)$.

Let X be the set of k-multisets of $[n]$.
Let C be a cyclic subgroup of S_n that is generated by the cycle $c = (1, 2, 3, \cdots, n)$.

Example

If $n = 3$ and $k = 2$, then $X = \{11, 22, 33, 12, 13, 23\}$.
$(1, 2, 3)23 = 31$.

Let X be the set of k-multisets of $[n]$.
Let C be a cyclic subgroup of S_n that is generated by the cycle $c = (1, 2, 3, \cdots, n)$.

Example

If $n = 3$ and $k = 2$, then $X = \{11, 22, 33, 12, 13, 23\}$.
$(1, 2, 3)23 = 31$.

Define $f(q)$ as $\binom{n+k-1}{k}_q$.

Let X be the set of k-multisets of $[n]$.
Let C be a cyclic subgroup of S_n that is generated by the cycle $c = (1, 2, 3, \cdots, n)$.

Example

If $n = 3$ and $k = 2$, then $X = \{11, 22, 33, 12, 13, 23\}$.
$(1, 2, 3)23 = 31$.

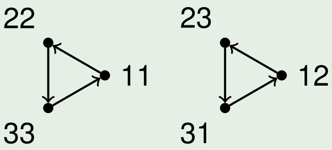
Define $f(q)$ as $\binom{n+k-1}{k}_q$.

Theorem

$(X, C, f(q))$ *defined as above exhibits the cyclic sieving phenomenon.*

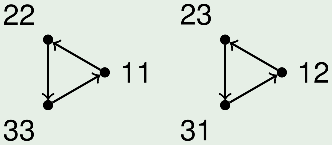
Example

Let $n = 3$ and $k = 2$.



Example

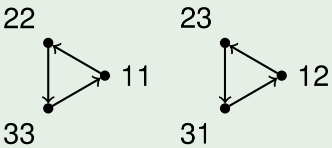
Let $n = 3$ and $k = 2$.



$f(q)$ is $\binom{3+2-1}{2}_q = 1 + q + 2q^2 + q^3 + q^4$.

Example

Let $n = 3$ and $k = 2$.

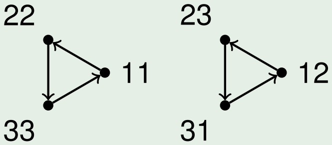


$f(q)$ is $\binom{3+2-1}{2}_q = 1 + q + 2q^2 + q^3 + q^4$.

- $|X^{id}| = 6 = f(\zeta^0)$.

Example

Let $n = 3$ and $k = 2$.

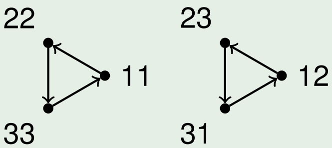


$f(q)$ is $\binom{3+2-1}{2}_q = 1 + q + 2q^2 + q^3 + q^4$.

- $|X^{id}| = 6 = f(\zeta^0)$.
- $|X^{(1,2,3)}| = 0 = f(\zeta^1)$.

Example

Let $n = 3$ and $k = 2$.



$f(q)$ is $\binom{3+2-1}{2}_q = 1 + q + 2q^2 + q^3 + q^4$.

- $|X^{id}| = 6 = f(\zeta^0)$.
- $|X^{(1,2,3)}| = 0 = f(\zeta^1)$.
- $|X^{(1,2,3)^2}| = 0 = f(\zeta^2)$.

Definition

Let V be a complex vector space. A group homomorphism $[\cdot] : G \to GL(V)$ is called representation.

Definition

Let V be a complex vector space. A group homomorphism $[\cdot] : G \to GL(V)$ is called representation.

When G acts on V, there is a natural choice of representation: $[g]v := gv$.

Definition

The character of a representation is $\chi : G \to \mathbb{C}$ such that $\chi(g) = \text{tr}[g]$.

Definition

The character of a representation is $\chi : G \to \mathbb{C}$ such that $\chi(g) = \mathrm{tr}[g]$.

Example

If $V = \mathbb{C}^3$ and S_3 acts on V by permuting the components, then the matrix form of $g = (12)$ in the standard basis is

$$[g]_B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\chi(g)$ is 1.

Define $\mathbb{C}X := \{c_1x_1 + c_2x_2 + \cdots + c_mx_m | x_i \in X\}$. $g = c^d$ acts on $\mathbb{C}X$.
We will evaluate $\chi(g)$ in two different basis.

Define $\mathbb{C}X := \{c_1x_1 + c_2x_2 + \cdots + c_mx_m | x_i \in X\}$. $g = c^d$ acts on $\mathbb{C}X$.
We will evaluate $\chi(g)$ in two different basis.

Method 1:
X is a standard basis of $\mathbb{C}X$.
The diagonal entry of $[g]_X$ is 1 if multiset $M \in X$ is fixed by g and 0 otherwise. Therefore, $\chi(g) = \text{tr}[g]_X = |X^g|$.

Example

If $n = 3$ and $k = 2$ as before and $g = c^1 = (1, 2, 3)$, then

$$\begin{aligned} g(11) &= 22, \quad g(22) = 33, \quad g(33) = 11, \\ g(12) &= 23, \quad g(23) = 31, \quad g(31) = 12. \end{aligned}$$

So $[g]_{\{11,22,33,12,23,31\}}$ is equal to

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Therefore, $\chi(g) = 0 = |X^g|$.

Method 2:

Let $c = (1, 2, 3, \cdots, n) \in S_n$. The characteristic polynomial of c is $x^n - 1$, which has n distinct roots: $1, \zeta, \zeta^2, \cdots, \zeta^{n-1}$.

Method 2:

Let $c = (1, 2, 3, \cdots, n) \in S_n$. The characteristic polynomial of c is $x^n - 1$, which has n distinct roots: $1, \zeta, \zeta^2, \cdots, \zeta^{n-1}$.

So there must be a basis $B = \{b_0, b_1, \cdots, b_{n-1}\}$ of $\mathbb{C}[n]$ such that the representation of c in $GL(\mathbb{C}[n])$ is diagonalized to $\text{diag}(1, \zeta, \cdots, \zeta^{n-1})$ by B, i.e. $c(b_i) = \zeta^i b_i$.

Method 2:
Let $c = (1, 2, 3, \cdots, n) \in S_n$. The characteristic polynomial of c is $x^n - 1$, which has n distinct roots: $1, \zeta, \zeta^2, \cdots, \zeta^{n-1}$.

So there must be a basis $B = \{b_0, b_1, \cdots, b_{n-1}\}$ of $\mathbb{C}[n]$ such that the representation of c in $GL(\mathbb{C}[n])$ is diagonalized to $\text{diag}(1, \zeta, \cdots, \zeta^{n-1})$ by B, i.e. $c(b_i) = \zeta^i b_i$.

$[c^d]_B$ is $\text{diag}(1^d, \zeta^d, \cdots, (\zeta^{n-1})^d)$.

Method 2:

Let $c = (1, 2, 3, \cdots, n) \in S_n$. The characteristic polynomial of c is $x^n - 1$, which has n distinct roots: $1, \zeta, \zeta^2, \cdots, \zeta^{n-1}$.

So there must be a basis $B = \{b_0, b_1, \cdots, b_{n-1}\}$ of $\mathbb{C}[n]$ such that the representation of c in $GL(\mathbb{C}[n])$ is diagonalized to $\mathrm{diag}(1, \zeta, \cdots, \zeta^{n-1})$ by B, i.e. $c(b_i) = \zeta^i b_i$.

$[c^d]_B$ is $\mathrm{diag}(1^d, \zeta^d, \cdots, (\zeta^{n-1})^d)$.

The set of k-multisets of B is an another basis for $\mathbb{C}X$.

Example

$$g(b_0b_0) = (\zeta^0 b_0)(\zeta^0 b_0), \quad g(b_1b_1) = (\zeta^1 b_1)(\zeta^1 b_1),$$
$$g(b_2b_2) = (\zeta^2 b_2)(\zeta^2 b_2), \quad g(b_0b_1) = (\zeta^0 b_0)(\zeta^1 b_1),$$
$$g(b_1b_2) = (\zeta^1 b_1)(\zeta^2 b_2), \quad g(b_2b_0) = (\zeta^2 b_2)(\zeta^0 b_0)$$

So $[g]_{\{b_0b_0, b_1b_1, b_2b_2, b_0b_1, b_1b_2, b_2b_0\}}$ is equal to

$$\begin{bmatrix} \zeta^0 \cdot \zeta^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta^1 \cdot \zeta^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta^2 \cdot \zeta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta^0 \cdot \zeta^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta^1 \cdot \zeta^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta^2 \cdot \zeta^0 \end{bmatrix}$$

$\chi(g) = \zeta^0 \cdot \zeta^0 + \zeta^1 \cdot \zeta^1 + \zeta^2 \cdot \zeta^2 + \zeta^0 \cdot \zeta^1 + \zeta^1 \cdot \zeta^2 + \zeta^2 \cdot \zeta^0 = 1 + \zeta + 2\zeta^2 + \zeta^3 + \zeta^4$.

Theorem

Let X be the set of k-subsets of $[n]$, then

$$\left(X, \langle (1, 2, \cdots, n) \rangle, \binom{n}{k}_q \right)$$

exhibits cyclic sieving phenomenon.

Definition

The nth Catalan number, $C_n = \frac{1}{n+1}\binom{2n}{n}$, is the number of expressions containing n pairs of balanced brackets.

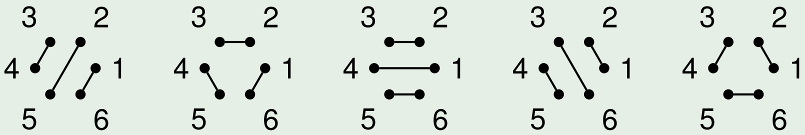
Example

1	2	3	4	5	6	7	8	is balanced.
(	)	(	(	)	(	)	)	
1	2	3	4	5	6	7	8	is not.
(	)	)	(	(	)	)	)	

Example

$C_3 = 5.$

1 2 3 4 5 6	1 2 3 4 5 6	1 2 3 4 5 6	1 2 3 4 5 6	1 2 3 4 5 6
((( )))	(( ) ( ))	(( )) ( )	( ) (( ))	( ) ( ) ( )



Rotation by π/n is an action on noncrossing matchings.

Rotation by π/n is an action on noncrossing matchings.

Theorem

Let X be the set of noncrossing matchings and R be the rotation by π/n. Then

$$\left(X, \langle R \rangle, \frac{1}{[2n+1]_q} \binom{2n}{n}_q \right)$$

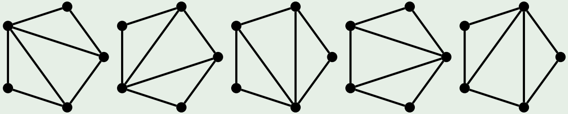
exhibits cyclic sieving phenomenon.

The number of triangulations of a regular $n + 2$-gon is also C_n.

The number of triangulations of a regular $n+2$-gon is also C_n.

Example

$C_3 = 5$



Theorem

Let X be the set of triangulations of regular $n+2$-gon and R be the rotation by $2\pi/(n+2)$. Then

$$\left(X, \langle R\rangle, \frac{1}{[2n+1]_q}\binom{2n}{n}_q\right)$$

exhibits cyclic sieving phenomenon.